DHANAMANJURI UNIVERSITY
DECEMBER-2025

Name of Programme : M.A./M.Sc. Mathematics
Semester : 1st
Paper Code : MAT-502
Paper Title : Real Analysis-I
Full Marks : 80
Pass Marks : 32 **Duration: 3 hours**

The figures in the margin indicate full marks for the questions.

Answers all the questions:

1. Answer any three from the following questions: $10 \times 3 = 30$

a) i) Let P_1 and P_2 be any two partitions of $[a, b]$. Show that $L(P_1, f, \alpha) \leq U(P_2, f, \alpha)$

ii) Show that $f \in \mathcal{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$, there exists a partition P of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

b) i) If f is monotonic on $[a, b]$, and α is continuous and monotonically increasing on $[a, b]$, then show that $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

ii) Suppose f is bounded on $[a, b]$, f has only finitely many points of discontinuity on $[a, b]$, α is monotonically increasing on $[a, b]$, and α is continuous at every point at which f is discontinuous. Show that $f \in \mathcal{R}(\alpha)$ on $[a, b]$.

c) i) If $a < s < b$, f is bounded on $[a, b]$, f is continuous at s, and $\alpha(x) = I(x - s)$, then show that $\int_a^b f d\alpha = f(s)$.

ii) Suppose $c_n \geq 0$ for $n = 1,2,3, \ldots$, $\sum_{n=1}^{\infty} c_n$ converges, $\{s_n\}$ is a sequence of distinct points in (a, b), and $\alpha(x) = \sum_{n=1}^{\infty} c_n I(x - s_n)$. Let f be continuous on $[a, b]$. Show that $\int_a^b f d\alpha = \sum_{n=1}^{\infty} c_n f(s_n)$.

d) i) Let f be a bounded function defined on $[-1,1]$, and let β be defined on $[-1,1]$ as follows

$$\beta(x) = \begin{cases} 0 & if\ x < 0 \\ \frac{1}{2} & if\ x = 0 \\ 1 & if\ x > 0 \end{cases}$$

Show that $f \in \mathcal{R}(\beta)$ if and only if f is continuous at $x = 0$.

ii) If f be bounded in $[a, b]$ and c be a constant, then evaluate $\int_a^b f(x)dc$.

e) i) When is a curve said to be rectifiable?

ii) If γ is a continuously differentiable curve defined on $[a, b]$, then show that γ is rectifiable and $\Lambda(\gamma) = \int_a^b |\gamma'(t)| dt$.

2. nswer any three from the following questions: $10 \times 3 = 30$

a) Define rearrangement of a series. If $\sum_{n=1}^{\infty} a_n$ converges absolutely to A, then show that any rearrangement $\sum_{n=1}^{\infty} a_n'$ of $\sum_{n=1}^{\infty} a_n$ converges absolutely to A.

b) Show that the sequence $\{f_n\}$, where $f_n(x) = nx(1-x)^n$ is not uniformly convergent on $[0,1]$.

c) State and prove Weierstrass's M-test for uniform convergence of an infinite series of functions. Hence show that $\sum_{n=1}^{\infty} \frac{nx^2}{n^3+x^3}$ is uniformly convergent on $[0,1]$.

d) Suppose $\{f_n\}$ is a sequence of functions differentiable on $[a, b]$ such that $\{f_n(x_0)\}$ converges for some point x_0 on $[a, b]$. If $\{f_n'\}$ converges uniformly on $[a, b]$, then show that $\{f_n\}$ converges uniformly on $[a, b]$, to a function f, and $f'(x) = \lim_{n\to\infty} f_n'(x)$.

e) State and prove Dirichlet's test for uniform convergence of series of functions.

3. Answer any two from the following questions: $10 \times 2 = 20$

a) Let Ω be the set of all invertible linear operators on $\mathbb{R}^n$. Prove that

i) If $A \in \Omega$, $B \in L(\mathbb{R}^n)$, and $\|B - A\|.\|A^{-1}\| < 1$, then $B \in \Omega$.

ii) Ω is an open subset of $L(\mathbb{R}^n)$, and the mapping $A \to A^{-1}$ is continuous on Ω.

b) Suppose f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}$. Show that $f \in C'(E)$ if and only if the partial derivatives $D_j f$ exist and are continuous on E for $1 \leq j \leq n$.

c) State and prove Implicit function theorem.

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